Q 2 from HW 1:

Let $f, g \in M_{m\times n}(\mathbb{R})$, and assume that they are periodically extended.

Prove that $f * g = g * f$.

Proof: Let $0 \le a \le m-1$, $0 \le b \le n-1$

$$\left(g * f(a,b) = \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} g(k,l) f(a-k, b-l) \right)$$

Then, $f * g(a,b) = \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} f(s,t) g(a-s, b-t)$

(Let $k = a-s$, $l = b-t$)

$$= \sum_{k=a-m+1}^{a} \sum_{l=b-n+1}^{b} f(a-k, b-l) g(k,l)$$

$$= \left(\sum_{k=0}^{a} \sum_{l=0}^{b} + \underbrace{\sum_{k=a-m+1}^{-1} \sum_{l=b-n+1}^{-1}}_{(*)} + \underbrace{\sum_{k=0}^{a} \sum_{l=b-n+1}^{-1}}_{(**)} + \underbrace{\sum_{k=a-m+1}^{-1} \sum_{l=0}^{b}}_{(***)} \right) f(a-k, b-l) g(k,l)$$

Let $p = k+m$, $q = l+n$

$$(*) = \sum_{p=a+1}^{m-1} \sum_{q=b+1}^{n-1} g(p-m, q-n) f(a-p+m, b-q+n)$$

$$= \sum_{p=a+1}^{m-1} \sum_{q=b+1}^{n-1} g(p,q) f(a-p, b-q)$$

By periodicity of f and g.

Let $q = l+n$

$$(**) = \sum_{k=0}^{a} \sum_{q=b+1}^{n-1} f(a-k, b-q+n) g(k, q-n)$$

$$= \sum_{k=0}^{a} \sum_{q=b+1}^{n-1} f(a-k, b-q) g(k,q)$$

Similarly, let $p = k+m$

$$(***) = \sum_{p=a+1}^{m-1} \sum_{q=0}^{b} f(a-p, b-l) g(p,l)$$

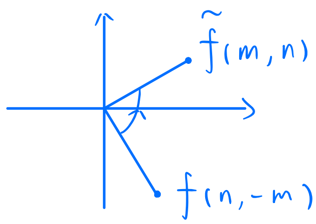
$$= \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} g(k,l) f(a-k, b-l)$$

$$= g * f(a,b)$$

Q: Let $f \in M_{M\times N}(\mathbb{R})$ be periodically extended.

Prove that $\hat{\tilde{f}} = \tilde{\hat{f}}$, where $\hat{f}$ is DFT of f,

and $\tilde{f}(m,n) = f(n,-m)$ (i.e., counter-clockwise rotation by 90°)



Proof: $\tilde{\hat{f}}(k,l) = \hat{f}(l,-k) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\, e^{-2\pi j\left(\frac{lm}{M} - \frac{kn}{N}\right)}$

$$\hat{\tilde{f}}(k,l) = \frac{1}{MN}\sum_{p=0}^{N-1}\sum_{q=0}^{M-1} \tilde{f}(p,q)\, e^{-2\pi j\left(\frac{kp}{N} + \frac{lq}{M}\right)}$$

$$= \frac{1}{MN}\sum_{p=0}^{N-1}\sum_{q=0}^{M-1} f(q,-p)\, e^{-2\pi j\left(\frac{kp}{N} + \frac{lq}{M}\right)}$$

Let $m = q$, $n' = -p$

$$= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n'=1-N}^{0} f(m,n')\, e^{-2\pi j\left(\frac{lm}{M} - \frac{kn'}{N}\right)}$$

Let $n = n' + N$

$$= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n-N)\, e^{-2\pi j\left(\frac{lm}{M} - \frac{kn}{N}\right) - 2\pi kj}$$

$$= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\, e^{-2\pi j\left(\frac{lm}{M} - \frac{kn}{N}\right)}$$

(Since $f(m,n) = f(m,n-N)$ and $e^{2\pi kj} = 1$)

Another proof via inverse transform:

It suffices to show $\mathrm{iDFT}(\tilde{\hat{f}}) = \tilde{f}$ $\qquad \tilde{f}(m,n) = f(n,-m)$

$$\mathrm{iDFT}(\tilde{\hat{f}})(m,n) = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \tilde{\hat{f}}(k,l)\, e^{2\pi j \left(\frac{km}{N} + \frac{ln}{M}\right)}$$

$$= \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \hat{f}(l,-k)\, e^{2\pi j \left(\frac{km}{N} + \frac{ln}{M}\right)}$$

$$= \frac{1}{MN} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \left(\sum_{s=0}^{M-1} \sum_{t=0}^{N-1} f(s,t)\, e^{-2\pi j \left(\frac{ls}{M} - \frac{kt}{N}\right)} \right) e^{2\pi j \left(\frac{km}{N} + \frac{ln}{M}\right)}$$

$$= \frac{1}{MN} \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} f(s,t) \underbrace{\sum_{k=0}^{N-1} e^{2\pi j \frac{k(t+m)}{N}}}_{(*)} \underbrace{\sum_{l=0}^{M-1} e^{2\pi j \frac{l(n-s)}{M}}}_{(**)}$$

$$(*) = \sum_{k=0}^{N-1} e^{2\pi j \frac{k(t+m)}{N}} = \begin{cases} \dfrac{1 - e^{2\pi j \cdot k(t+m)}}{1 - e^{2\pi j \frac{t+m}{N}}} = 0, & \text{if } t+m \notin N\mathbb{Z} \\ N, & \text{if } t+m \in N\mathbb{Z} \end{cases}$$

Similarly,

$$(**) = \begin{cases} 0, & \text{if } n-s \notin M\mathbb{Z} \\ M, & \text{if } n-s \in M\mathbb{Z} \end{cases}$$

$$= \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} f(s,t)\, \mathbb{1}_{N\mathbb{Z}}(t+m)\, \mathbb{1}_{M\mathbb{Z}}(n-s)$$

$$= f(n, -m+N)$$

$$= f(n,-m) \qquad \text{By Periodicity}$$

$$= \tilde{f}(m,n)$$

Understanding low pass and high pass filters.

We start from 1D DFT.

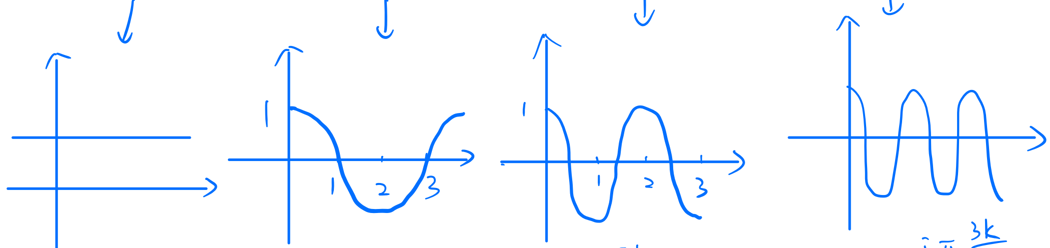
Given $f(k)$: $k = 0, 1, \cdots\cdots, M-1$.

The DFT: $\tilde{f}(t) = \frac{1}{M} \sum_{k=0}^{M-1} f(k)\, e^{-2\pi j \frac{kt}{M}}$

iDFT: $f(k) = \sum_{t=0}^{M-1} \tilde{f}(t)\, e^{2\pi j \frac{kt}{M}}$.

Take $M = 4$ as an example:

$f(k) = \tilde{f}(0) + \tilde{f}(1) e^{2\pi j \frac{k}{4}} + \tilde{f}(2) e^{2\pi j \cdot \frac{k}{2}} + \tilde{f}(3) e^{2\pi j \frac{3k}{4}}$



$e^{j\frac{\pi k}{2}} = \cos(\frac{\pi}{2}k) + i\sin(\frac{\pi}{2}k)$

$e^{\pi j k} = \cos(\pi k) + i\sin(\pi k)$

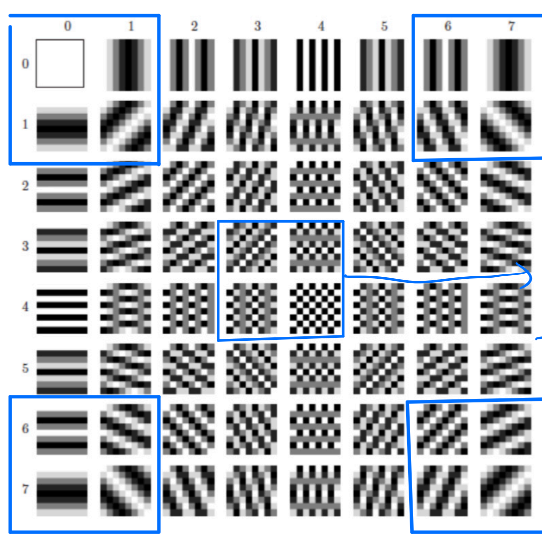
$e^{j\pi\frac{3k}{2}} = \cos(\frac{3\pi}{2}k) + i\sin(\frac{3\pi}{2}k)$

low frequency $\longrightarrow$ high frequency

2D case:

$g(m,n) = \sum_{k}\sum_{\ell} \tilde{g}(k,\ell)\, e^{2\pi j(\frac{mk}{M} + \frac{n\ell}{n})}$

$g = U\tilde{g}U = \sum_{k}\sum_{\ell} \hat{g}_{k\ell} \underbrace{\vec{W_k}\,\vec{W_\ell}^T}_{\text{eigenimage}}$



high frequency

low frequency